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Anosov flows with smooth foliations and rigidity of geodesic flows on three-dimensional manifolds of negative curvature

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Abstract. We consider Anosov flows on a 5-dimensional smooth manifold V that possesses an invariant symplectic form (transverse to the flow) and a smooth invariant probability measure λ. Our main technical result is the following: If the Anosov foliations are C^∞, then either (1) the manifold is a transversely locally symmetric space, i.e. there is a flow-invariant C^∞ affine connection ∇ on V such that $\nabla R \equiv 0$, where R is the curvature tensor of ∇, and the torsion tensor T only has nonzero component along the flow direction, or (2) its Oseledec decomposition extends to a C^∞ splitting of TV (defined everywhere on V) and for any invariant ergodic measure μ, there exists $\chi_\mu > 0$ such that the Lyapunov exponents are $-2\chi_\mu$, $-\chi_\mu$, 0, χ_μ, and $2\chi_\mu$, μ-almost everywhere.

As an application, we prove: Given a closed three-dimensional manifold of negative curvature, assume the horospheric foliations of its geodesic flow are C^∞. Then, this flow is C^∞ conjugate to the geodesic flow on a manifold of constant negative curvature.

1. *Introduction*

Let V be a compact C^∞ manifold without boundary (we will use 'C^∞' and 'smooth' interchangeably) and let $\varphi_t : V \to V$, $t \in \mathbb{R}$, be a C^∞ *Anosov flow* on V. Denote by $TV = E^+ \oplus E^- \oplus E^0$ the φ_t-invariant splitting of the tangent bundle of V into the distributions E^+ of expanding vectors, E^- of contracting vectors, and the direction E^0 spanned by the vector field

$$\dot\varphi := \frac{d}{dt}\bigg|_{t=0} \varphi_t.$$

We recall that there are numbers $a > 0$ and $b \geq 1$ such that, for all $t \geq 0$,

$$\|D\varphi_{-t}|_{E^+}\| \leq b \cdot e^{-at} \quad \|D\varphi_t|_{E^-}\| \leq b \cdot e^{-at}. \tag{0}$$

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Define $E := E^+ \oplus E^-$ and assume that E possesses a symplectic form Ω, which is C^∞ and flow-invariant. In this case E^+ and E^- have equal dimension and are Lagrangian. We may also consider an Anosov diffeomorphism on a symplectic manifold V, $\varphi: V \to V$, in which case $TV = E = E^+ \oplus E^-$.

We also assume that V is equipped with a smooth (C^∞) φ_t-invariant probability measure λ. Then φ_t is ergodic with respect to λ.

The most natural case in which the above conditions are satisfied is that of a *contact flow*, i.e. when $\dim V = 2m+1$ and there exists a flow-invariant 1-form θ such that $\nu = \theta \wedge (d\theta)^m \neq 0$. Then one can define $\Omega = d\theta$ and $\lambda = |\nu|$. If a contact flow is Anosov, then automatically $E^+ \oplus E^- = \operatorname{Ker} \theta$ and $\Omega(E^0, \cdot) = 0$.

Let us denote $E^{+0} := E^+ \oplus E^0$, $E^{-0} := E^- \oplus E^0$. The distributions E^+, E^-, E^{+0}, E^{-0} are all integrable. Let W^+, W^-, W^{+0}, W^{-0} denote the corresponding foliations. W^+ and W^- are Lagrangian foliations with respect to Ω.

In this context, it is possible to define a flow-invariant affine connection ∇ on V (see § 2), which was first used by M. Kanai in [**K**]. ∇ is transversely torsion-free, meaning that for any vector fields ξ, η on V, $T(\xi, \eta) \in E^0$ (T is the torsion tensor of ∇). The curvature tensor will be denoted by R.

We say that (V, ∇) is an *affine locally symmetric space on transversals* if $\nabla R \equiv 0$. If for some cover $\tilde{V}$ of V the space P of orbits of the lifted flow to $\tilde{V}$, with the quotient topology, is a smooth manifold, this condition means that P (with the affine connection induced by ∇) is an affine locally symmetric space.

Our main result is:

THEOREM 1. *Let $\varphi_t: V \to V$ be an Anosov flow as above and assume that the dimension of V is 5. Assume that E^+ and E^- are C^∞. Then, either*

(1) *(V, ∇) is an affine locally symmetric space on transversals or*

(2) *There exist C^∞ φ_t-invariant line fields L_i^ε, $\varepsilon \in \{+,-\}$, $i \in \{1, 2\}$, defined everywhere on V, such that $E^\varepsilon = L_1^\varepsilon \oplus L_2^\varepsilon$ and for any ergodic φ_t-invariant measure μ, there is $\chi_\mu > 0$ such that for μ-almost everywhere $v \in V$ and for every $0 \neq \xi \in L_i^\varepsilon(v)$,*

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|(D\varphi_t)_v \xi\| = \varepsilon i \chi_\mu \quad \varepsilon \in \{+, -\},\ i \in \{1, 2\},$$

(here, $\|\cdot\|$ denotes any Riemannian norm on TV).

In other words, the Oseledec decomposition of $\{\varphi_t\}$ is smooth and the Lyapunov exponents are equal to $-2\chi_\mu$, $-\chi_\mu$, 0, χ_μ, $2\chi_\mu$, μ-almost everywhere.

Moreover, there exists $\varepsilon \in \{+,-\}$ for which $L_1^\varepsilon \subset [L_2^\varepsilon, L_1^{-\varepsilon}]$, on a nontrivial invariant open set.

The same results hold, with the obvious rephrasing, for symplectic Anosov diffeomorphisms on compact four-dimensional manifolds.

We give an application of Theorem 1 to the problem of rigidity of geodesic flows on 3-dimensional Riemannian manifolds of negative curvature.

Let M be a compact C^∞ manifold without boundary, of dimension $n \geq 2$, with a Riemannian metric σ of negative sectional curvature. The geodesic flow $\varphi_t: V \to V$ on the unit tangent bundle V of M is a contact Anosov flow, and V is foliated by the horospheric foliations W^+ (resp. W^{+0}), the strong (resp. weak) unstable foliation and W^- (resp. W^{-0}), the strong (resp. weak) stable foliation.

In [**K**], M. Kanai shows that if the stable and unstable foliations W^+ and W^- are C^∞ and the sectional curvature K of M satisfies

$$-9/4 < K \leq -1$$

then the geodesic flow on V is C^∞ isomorphic to the geodesic flow for a metric of constant negative curvature. In [**F–K**], the above condition is improved to the optimal one: $-4 < K \leq -1$.

By combining Theorem 1 with a fact that appears in the proof of Theorem 4.1 of [**K**], we show that the pinching assumption on K is not necessary if dim $M = 3$.

THEOREM 2. *Let M be a compact, boundaryless, C^∞ Riemannian manifold of dimension* 3 *whose sectional curvature K is strictly negative. Assume that one of the horospheric foliations W^+ or W^-, in the unit tangent bundle V of M, is C^∞. Then, the geodesic flow on V is C^∞ isomorphic to the geodesic flow for a metric of constant negative curvature.*

We observe that, in the case of geodesic flows, the smoothness of, say, W^- readily implies the smoothness of W^+. This is because there exists a diffeomorphism of V, the *flip map* $J: v \mapsto -v$, which interchanges W^+ and W^-.

Remarks. (i) Two previous proofs of Theorem 2 presented in the preprints *Rigidity of Geodesic Flows on Negatively Curved Compact* 3*-Manifolds*, by the second author, and *Rigidity of Geodesic Flows on Negatively Curved Manifolds of Dimensions* 3 *and* 4, contain gaps, although the main line of argument is correct and is carried out in the present paper. Another reason for restructuring the paper is our desire to separate the main technical result for general Anosov flows (Theorem 1), which essentially belongs to smooth dynamics, from the application to geodesic flows. The interested reader will find other applications of Theorem 1 to Anosov diffeomorphisms in [**Fl–K**].

(ii) All the results proven here require only finite smoothness of the Anosov foliations. The exact degree of smoothness needed is, however, much greater than the optimal one (which is presumably C^2). Therefore we did not pay greater attention to this issue. We note that we have used Sard's theorem in Lemma 6 and that Kanai's results in [**K**] are formulated for C^∞ foliations (although finite smoothness suffices there, too).

(iii) After this paper was written, the first author extended the result of Theorem 2 to negatively curved manifolds of arbitrary odd dimension [**F**].

2. *The Kanai connection*

Assume the setting defined in the introduction, prior to Theorem 1. To avoid repetitions, we will restrict ourselves to the case of flows.

Denote by $\pi^\varepsilon: TV \to E^\varepsilon$, $\varepsilon = +, -$, or 0, the natural projections, and assume that the bundles E^ε are differentiable of class C^r, $r \geq 1$. The *Kanai connection* is an affine connection ∇ on V such that (i) $\nabla\Omega \equiv 0$, (ii) $\nabla\pi^\varepsilon \equiv 0$, for $\varepsilon = +, -, 0$, (iii) ∇

is transversely torsion-free, i.e. $\pi^\varepsilon T = 0$, $\varepsilon = +, -$, (iv) $\nabla\dot\varphi \equiv 0$, (v) $\nabla_{\dot\varphi} = \mathfrak{L}_{\dot\varphi}$ = the Lie derivative along the flow.

It is not difficult to verify that property (ii) is equivalent to the following one: (ii′) If ξ^ε is a vector field in E^ε and η is any vector field on V, then $\nabla_\eta \xi^\varepsilon \in E^\varepsilon$, for $\varepsilon = +, -$, or 0.

LEMMA 1. *There exists a unique affine connection ∇ on V satisfying* (i)–(v). *∇ is φ_t-invariant and is of class C^{r-1} if the bundles E^+ and E^- are of class C^r.*

Proof. (See also [**K**] and [**F–K**].) We first show that there exists a unique covariant derivative of vector fields in E, along vectors in E, satisfying the properties (i)–(iii).

Define $c = \pi^+ - \pi^-$ and let $g = \Omega(\cdot, c\,\cdot)$. g is a bilinear nondegenerate symmetric form on E and one can define the corresponding Levi-Civita connection, the unique torsion-free connection ∇' with respect to which g is a parallel tensor field, i.e. $\nabla' g \equiv 0$. Then note that $\nabla' c \equiv 0$ is equivalent to $\nabla' \pi^\varepsilon \equiv 0$, $\varepsilon = +, -$ (we observe that $2 \cdot \pi^\varepsilon = \text{Identity} + \varepsilon \cdot c$). We will show that the latter property is equivalent to ∇' preserving the subbundles E^ε, $\varepsilon = +, -$.

Given vector fields ξ, η, ν in E, we have (see, e.g. [**KN**] v. I, p. 36, and [**H**], p. 48)

$$\text{(a)}\quad 2g(\nabla'_\xi \eta, \nu) = \xi g(\eta, \nu) + \eta g(\xi, \nu) - \nu g(\xi, \eta) - g([\eta, \nu], \xi) - g([\xi, \nu], \eta) + g([\xi, \eta], \nu)$$

$$\text{(b)}\quad 0 = 3d\Omega(\xi, \eta, \nu) = \xi\Omega(\eta, \nu) - \eta\Omega(\xi, \nu) + \nu\Omega(\xi, \eta) - \Omega([\eta, \nu], \xi) + \Omega([\xi, \nu], \eta) - \Omega([\xi, \eta], \nu).$$

Given vector fields ξ^ε, η^ε, ν^ε in E^ε, $\varepsilon = +, -$, it follows from (a), (b), the integrability of E^ε, and the identities $\Omega(E^\varepsilon, E^\varepsilon) \equiv 0$ and $g(E^\varepsilon, E^\varepsilon) \equiv 0$, that

$$g(\nabla'_{\xi^\varepsilon} \eta^\varepsilon, \nu^\varepsilon) = 0 \quad \text{hence } \nabla'_{\xi^\varepsilon} \eta^\varepsilon \in E^\varepsilon \text{ and}$$

$$g(\nabla'_{\xi^\varepsilon} \eta^{-\varepsilon}, \nu^{-\varepsilon}) = \varepsilon/2\; d\Omega(\xi^\varepsilon, \eta^{-\varepsilon}, \nu^{-\varepsilon}) = 0 \quad \text{hence } \nabla'_{\xi^\varepsilon} \eta^{-\varepsilon} \in E^{-\varepsilon}.$$

It also follows from a simple computation that if $\nabla' c \equiv 0$ and $\nabla' g \equiv 0$ then $\nabla' \Omega \equiv 0$. Therefore ∇' satisfies (i), (ii), and (iii).

Given arbitrary vector fields $\xi = \xi_1 + f\dot\varphi$ and $\eta = \eta_1 + h\dot\varphi$ for real functions f and h, and vector fields ξ_1, η_1 in E, define

$$\nabla_\xi \eta = \nabla'_{\xi_1} \eta_1 + f \mathfrak{L}_{\dot\varphi} \xi + (\xi_1 h)\dot\varphi.$$

It is not difficult to check that ∇ so defined is the unique affine connection on V that satisfies (i)–(v). Moreover the connection $\varphi_t^* \nabla$, defined by $(\varphi_t^* \nabla)_\xi \eta :=$ $D\varphi_{-t} \nabla_{D\varphi_t \xi} D\varphi_t \eta$, also can be shown to have the same properties. By uniqueness, we must have $\nabla = \varphi_t^* \nabla$.

By computing the Christoffel symbols of ∇, one readily sees that ∇ is C^{r-1} if the foliations are C^r. □

Let g be the symmetric nondegenerate bilinear form on E introduced in the proof of Lemma 1: $g = \Omega(\cdot, c\,\cdot)$, $c = \pi^+ - \pi^-$. Denote by R the curvature tensor of the Kanai connection and consider the (0, 4)-tensor field $\check R := g(R(\cdot, \cdot)\cdot, \cdot)$. Its covariant derivative $\omega := \nabla \check R$ is a (0, 5)-tensor field and we have $\omega \equiv 0$ if and only if $\nabla R \equiv 0$, as can be easily verified.

LEMMA 2. (1) ω and $\check{R}$ are *φ_t-invariant tensor fields.*

(2) *If* $\omega_v \neq 0$ *for some* $v \in V$, *there exist* $\varepsilon = +$ *or* $-$, $\xi_1, \xi_3, \xi_5 \in E^{\varepsilon}(v)$, *and* $\xi_2, \xi_4 \in E^{-\varepsilon}(v)$, *such that* $\omega_v(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \neq 0$.

(3) *For any* $v \in V$, $\varepsilon \in \{+, -\}$ *and vectors* $\xi_1, \xi_3, \xi_5 \in E^{\varepsilon}(v)$, $\xi_2, \xi_4 \in E^{-\varepsilon}(v)$, *we have*

$$\omega_v(\xi_1, \ldots, \xi_5) = \omega_v(\xi_{\mu(1)}, \ldots, \xi_{\mu(5)}),$$

where μ *is any permutation of* $\{1, \ldots, 5\}$ *such that* $\mu = \mu_1 \circ \mu_2$, *a product of a permutation* μ_1 *of* $\{1, 3, 5\}$ *and a permutation* μ_2 *of* $\{2, 4\}$. *Moreover, for a permutation* μ *such that* $\mu(1) = 1$, *we have* $\check{R}_v(\xi_2, \ldots, \xi_5) = \check{R}_v(\xi_{\mu(2)}, \ldots, \xi_{\mu(5)})$.

Proof. (1) follows naturally from the φ_t-invariance of ∇ and g. In order to show the other properties, we need to consider the algebraic symmetries of ω. First, let us observe that $\check{R}_v(\xi_1, \xi_2, \xi_3, \xi_4) = 0$ whenever ξ_1 and ξ_2, or ξ_3 and ξ_4 belong to the same subbundle E^{ε} for $\varepsilon = +$ or $-$. In fact, as $R(\xi_1, \xi_2)E^{\varepsilon} \subset E^{\varepsilon}$ and E^{ε} is a Lagrangian subbundle, we have $\check{R}_v(\xi_1, \ldots, \xi_4) = \varepsilon\Omega(R_v(\xi_1, \xi_2)\xi_3, \xi_4) = 0$, whenever ξ_3 and ξ_4 belong to the same space E^{ε}. On the other hand, it is well known that the curvature tensor of an affine connection associated to an indefinite metric satisfies $\check{R}_v(\xi_1, \xi_2, \xi_3, \xi_4) = \check{R}(\xi_3, \xi_4, \xi_1, \xi_2)$, so that the same property holds for the first pair.

$\check{R}$ also satisfies the following symmetries, true for any curvature tensor with an indefinite metric. We use the abbreviation (1 2 3 4) for $\check{R}(\xi_1, \xi_2, \xi_3, \xi_4)$.

$$(1\ 2\ 3\ 4) = (3\ 4\ 1\ 2) = -(2\ 1\ 3\ 4) = -(1\ 2\ 4\ 3)$$

$$(1\ 2\ 3\ 4) + (2\ 3\ 1\ 4) + (3\ 1\ 2\ 4) = 0 \text{ (first Bianchi identity).}$$

Furthermore, we obtain, for

$$(1\ 2\ 3\ 4\ 5) := \omega(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5),$$

(i) (1 2 3 4 5) = 0, whenever ξ_2 and ξ_3, or ξ_4 and ξ_5 belong to the same subbundle E^{ε}, $\varepsilon = +$ or $-$.

(ii) (1 2 3 4 5) = −(1 3 2 4 5)

(iii) (1 2 3 4 5) = −(1 2 3 5 4)

(iv) (1 2 3 4 5) = (1 4 5 2 3)

(v) (1 2 3 4 5) + (1 3 4 2 5) + (1 4 2 3 5) = 0

(vi) (1 2 3 4 5) + (2 3 1 4 5) + (3 1 2 4 5) = 0 (second Bianchi identity).

The identities (i)–(v) are easily obtained from the corresponding properties of $\check{R}$ by using that $\nabla E^{\varepsilon} \subset E^{\varepsilon}$ and the formula

$$\omega_v(\xi_1, \ldots, \xi_5) = \xi_1 \check{R}(\tilde{\xi}_2, \ldots, \tilde{\xi}_5) - \sum_{i=2}^{5} \check{R}(\xi_2, \ldots, \nabla_{\xi_1}\tilde{\xi}_i, \ldots, \xi_5),$$

where $\xi_i \in E^{\delta}(v)$ and $\tilde{\xi}_i$ is any smooth vector field in E^{δ} which extends ξ_i near v. It is now easy to show that (i)–(vi) imply (2) and (3). □

3. *Some smooth ergodic theory*

Let $\|\cdot\|$ be any Riemannian norm on TV. If $v \in V$ and $\xi \in T_vV$, define

$$\chi^{\pm}(v, \xi) := \limsup_{t \to \pm\infty} \frac{1}{t} \log \|(D\varphi_t)_v \xi\|.$$

For each $v \in V$, $\chi^+(v, \cdot)$ assumes finitely many values on T_vV, say

$$\chi_1(v) < \chi_2(v) < \cdots < \chi_{s(v)}(v) \quad s(v) \le \dim T_vV.$$

Define

$$F_i(v) = \{\xi \in T_vV : \chi^+(v, \xi) \le \chi_i(v)\}.$$

$F_i(v)$ is a linear subspace of T_vV, and we have the filtration:

$$\{0\} = F_0(v) \subset F_1(v) \subset \cdots \subset F_{s(v)}(v) = T_vV.$$

The functions χ_i, s, and the filtration (F_i) are φ_t-invariant and measurable, as functions of $v \in V$.

Let μ be any φ_t-invariant Borel probability measure on V. According to the *Multiplicative Ergodic Theorem* of Oseledec, there exists a set Λ of full μ-measure such that for all $v \in \Lambda$ we have the linear decomposition

$$T_vV = \bigoplus_{j=1}^{s(v)} E_j(v)$$

and $\chi^{\pm}(v, \xi) \triangleq \lim_{t \to \pm\infty} t^{-1} \log \|(D\varphi_t)_v \xi\| = \chi_j(v)$, uniformly in $\xi \in E_j(v)$ such that $\|\xi\| = 1$. The subspaces $E_j(v)$, $j = 1, \ldots, s(v)$, depend measurably on v and are φ_t-invariant.

The following lemma was proved in **[F–K]**.

LEMMA 3. *Let τ be a continuous φ_t-invariant tensor field on V of type* $(0, r)$. *Let $v \in \Lambda$ and suppose that $\xi_i \in E_{l_i}(v)$, $i = 1, \ldots, r$, are vectors at v such that $\tau(\xi_1, \ldots, \xi_r) \neq 0$. Then*

$$\sum_{i=1}^{r} \chi_{l_i}(v) = 0.$$

Recall the setup given in § 1. In particular, we have $E = E^+ \oplus E^-$ and the symplectic form Ω on E. By applying Lemma 3 to the invariant tensor field Ω, we note that, if χ is a Lyapunov exponent of φ_t, then $-\chi$ is also a Lyapunov exponent. In this case, the Oseledec decomposition reads, for $v \in \Lambda$,

$$E^\varepsilon(v) = E_1^\varepsilon(v) \oplus \cdots \oplus E_r^\varepsilon(v),$$

where $\varepsilon = +$ or $-$, $r = (s-1)/2$, and $E_i^\varepsilon(v)$ is associated to the Lyapunov exponent $\varepsilon \cdot \chi_i(v)$. Moreover, since φ_t is Anosov, if η is a vector such that $\chi^+(v, \eta) = 0$, then $\eta \in E^0$.

We may also consider the filtration

$$\{0\} = F_0^+ \subset F_1^+ \subset \cdots \subset F_r^+ = E^+,$$

where F_i^+ is defined by $F_{i+r+1}(v) = F_i^+(v) \oplus E^-(v)$.

The foliations W^+ and W^{-0} (see § 1) are transverse to each other and have complementary dimensions. Assume they are of class C^r, $r \ge 1$. Given any pair of points v and $w \in V$ such that $v \in W^{-0}(w)$, we can find a neighbourhood $\mathscr{U}_v$ of v and $\mathscr{U}_w$ of w for which the *holonomy map*

$$\mathscr{H}_{wv} : u \in W^+(v) \cap \mathscr{U}_v \mapsto W^{-0}(u) \cap W^+(w) \in W^+(w) \cap \mathscr{U}_w$$

is a C^r-diffeomorphism. Denote by $H_{wv}(u)$: $E^+(u) \to E^+(\mathscr{H}_{wv}(u))$ the differential of $\mathscr{H}_{wv}$ at $u \in W^+(v) \cap \mathscr{U}_v$.

Notice that, for every $t \in \mathbb{R}$,

$$\varphi_t \circ \mathcal{H}_{wv} \circ \varphi_{-t} = \mathcal{H}_{\varphi_t(w)\varphi_t(v)},$$

since the foliations are φ_t-invariant. Hence

$$D\varphi_t \circ H_{wv} \circ D\varphi_{-t} = H_{\varphi_t(w)\varphi_t(v)}.$$

LEMMA 4. *Assume that the Anosov splitting $TV = E^+ \oplus E^- \oplus E^0$ is differentiable of class C^r, $r \geq 1$. Let v, $w \in V$, with $v \in W^{-0}(w)$. Then $s(v) = s(w)$, $\chi_i(v) = \chi_i(w)$ for $i = 1, \ldots, s$, and for each i*

$$H_{wv}(v)F_i^+(v) = F_i^+(w).$$

In particular, the filtration (F_i^+) is C^{r-1} along the leaves of W^{-0}.

Proof. If $v \in W^-(w)$, $\xi \in F_i^+(v)$, and $\xi' := H_{wv}(v)\xi$, then $\chi^+(w, \xi') = \chi^+(v, \xi)$. This is, in fact, true since

$$\chi^+(w, \xi') = \limsup_{t \to +\infty} \frac{1}{t} \log \|D\varphi_t H_{wv}(v)\xi\|$$

$$= \limsup_{t \to +\infty} \frac{1}{t} \log \|H_{\varphi_t(w)\varphi_t(v)} D\varphi_t \xi\|.$$

But $\limsup_{t \to +\infty} \operatorname{dist}(\varphi_t(w), \varphi_t(v)) = 0$, so that there are constants $0 < c$, c' such that for every $t \geq 0$ and $\eta \in E^+(\varphi_t(v))$, we have

$$c'\|\eta\| \leq \|H_{\varphi_t(w)\varphi_t(v)}\eta\| \leq c\|\eta\|.$$

Therefore

$$\chi^+(w, \xi') = \limsup_{t \to +\infty} \frac{1}{t} \log \|D\varphi_t \xi\| = \chi^+(v, \xi).$$

It remains to show that the negative (forward) exponents at v and w are the same. For that, let us consider the following map: Let $\xi \in F_i(v) \cap E^-(v)$ and $\xi^\perp = \{\eta \in E^+(v): \Omega(\xi, \eta) = 0\}$. Choose any $\nu \in E^+(v)$ so that $\Omega(\nu, \xi) = 1$. Then, there exists a unique $\xi' \in E^-(w)$ such that $(\xi')^\perp = H_{wv}\xi^\perp$ and $\Omega(H_{wv}\nu, \xi') = 1$. The correspondence

$$P_{wv} : \xi \in E^-(v) \mapsto \xi' \in E^-(w)$$

is well defined (i.e. independent of the choice of ν) and C^r (if the Anosov splitting is C^r). Clearly, the same argument used above for H_{wv} also applies to P_{wv}, so that

$$\chi^+(w, \xi') = \chi^+(v, \xi). \qquad \square$$

As a remark, we point out that the maps H and P that occur in the above proof are nothing but the parallel transport of vectors with respect to the invariant affine connection ∇.

4. *The Proof of Theorems* 1 *and* 2

In § 2, we introduced the invariant $(0, 5)$-tensor field w. If $w = 0$ we arrive at the alternative (1) of Theorem 1. Now we would like to consider the possibility $\omega \neq 0$.

Consider the set $\mathcal{A}^\varepsilon := \{v \in V: \omega_v(E^\varepsilon, E^{-\varepsilon}, E^\varepsilon, E^{-\varepsilon}, E^\varepsilon) \neq 0\}$, i.e. $v \in \mathcal{A}^\varepsilon$ if there are vectors $\xi_1, \xi_3, \xi_5 \in E^\varepsilon(v)$, $\xi_2, \xi_4 \in E^{-\varepsilon}(v)$ such that $\omega_v(\xi_1, \ldots, \xi_5) \neq 0$.

From now on, λ will denote a smooth invariant probability measure on V. Recall that φ_t is ergodic with respect to λ. In particular, every nontrivial invariant open set has full λ-measure.

According to Lemma 2 (2), $\omega \neq 0$ implies $\mathscr{A}^+$ or $\mathscr{A}^-$ (or both) are non-empty. $\mathscr{A}^\varepsilon$ is an open subset of V and, since ω and $E^\pm$ are flow-invariant, so is $\mathscr{A}^\varepsilon$. Therefore, if $\mathscr{A}^\varepsilon \neq \emptyset$, for some $\varepsilon \in \{+,-\}$, $\mathscr{A}^\varepsilon$ has full λ-measure.

Let $\delta \in \{+,-\}$ be such that $\mathscr{A}^\delta \neq \emptyset$, and define $\Lambda^\delta = \mathscr{A}^\delta \cap \Lambda$. Note that Λ^δ also has a full λ-measure.

LEMMA 5. *Assume the same conditions as in Theorem* 1 (*in particular,* $\dim E^\pm(v) = 2$, $v \in V$). *Suppose that* $\omega \not\equiv 0$. *Then for each* $v \in \Lambda^\delta$ (*see definition above*), *we have the Oseledec decomposition*

$$T_vV = E_2^-(v) \oplus E_1^-(v) \oplus E_0(v) \oplus E_1^+(v) \oplus E_2^+(v)$$

and Lyapunov exponents $-\chi_2(v) < -\chi_1(v) < 0 < \chi_1(v) < \chi_2(v)$. *Moreover, we have*

$$\omega_v(E_{i_1}^\delta, E_{i_2}^{-\delta}, E_{i_3}^\delta, E_{i_4}^{-\delta}, E_{i_5}^\delta) \neq 0 \qquad (*)$$

for at least one of the combinations of subscripts shown in Table 1 *and for no other possibility not given there* (*the exponents will, in each case, satisfy the relation given at the right hand side of the table*).

Table 1

	i_1	i_2	i_3	i_4	i_5	Resonance
I	1	2	1	1	1	$2\chi_1(v) = \chi_2(v)$
	1	1	1	2	1	
II	2	2	1	2	1	$2\chi_1(v) = \chi_2(v)$
	1	2	2	2	1	
	1	2	1	2	2	
III	1	2	1	2	1	$3\chi_1(v) = 2\chi_2(v)$

Proof. Lemma 3, applied to Ω, yields that χ is a Lyapunov exponent if and only if $-\chi$ is a Lyapunov exponent. This remark and the same lemma applied to

$$\omega_v(E_{i_1}^\delta, E_{i_2}^{-\delta}, E_{i_3}^\delta, E_{i_4}^{-\delta}, E_{i_5}^\delta) \neq 0, \; v \in \Lambda^\delta,$$

show that there are two different positive exponents and

$$\chi_{i_1} + \chi_{i_3} + \chi_{i_5} = \chi_{i_2} + \chi_{i_4} \quad i_s \in \{1, 2\}.$$

It is, now, easy to verify that the above table accounts for all the possibilities. □

The next lemma essentially follows from the arguments in [**F–K**].

LEMMA 6. *There exists a* φ_t*-invariant set* $\hat{\Lambda} \subset \Lambda^\delta$ *of full* λ*-measure such that for every* $v \in \hat{\Lambda}$, (*) *holds* (*see Lemma* 5) *only for possibility I of Table* 1.

Proof. To simplify the notation, assume that $\delta = -$. We begin by showing that case II in table 1 can only happen in a set of λ-measure zero. First, note the following. Let $v \in \Lambda^-$ be such that

$$\omega_v(\xi_2^-, \xi_2^+, \xi_1^-, \xi_2^+, \xi_1^-) \neq 0 \tag{$*_1$}$$

for vectors $\xi_i^\varepsilon \in E_i^\varepsilon$, $i = 1$ or 2, $\varepsilon = +$ or $-$. Since E_i^ε are one-dimensional, there is no loss of generality in using repeated arguments for ω in $(*_1)$, as we did, for example, in its second and fourth entries. Denote by Λ' the set of such v's. Let $\tilde{\xi}_i^\varepsilon \in E^\varepsilon$ be arbitrary smooth extensions of ξ_i^ε in a neighbourhood of v. We claim that

$$0 \neq \omega_v(\xi_2^-, \xi_2^+, \xi_1^-, \xi_2^+, \xi_1^-) = \xi_2^- \check{R}(\tilde{\xi}_2^+, \tilde{\xi}_1^-, \tilde{\xi}_2^+, \tilde{\xi}_1^-). \tag{$*_2$}$$

In general, we should have

$$\begin{aligned}\omega_v(\xi_2^-, \xi_2^+, \xi_1^-, \xi_2^+, \xi_1^-) &- \xi_2^- \check{R}(\tilde{\xi}_2^+, \tilde{\xi}_1^-, \tilde{\xi}_2^+, \tilde{\xi}_1^-)\\ &= -\check{R}(\nabla_{\xi_2^-}\tilde{\xi}_2^+, \xi_1^-, \xi_2^+, \xi_1^-) - \check{R}(\xi_2^+, \nabla_{\xi_2^-}\tilde{\xi}_1^-, \xi_2^+, \xi_1^-)\\ &\quad -\check{R}(\xi_2^+, \xi_1^-, \nabla_{\xi_2^-}\tilde{\xi}_2^+, \xi_1^-) - \check{R}(\xi_2^+, \xi_1^-, \xi_2^+, \nabla_{\xi_2^-}\tilde{\xi}_1^-).\end{aligned}$$

If, say, the first term on the right hand side of the above identity is not zero, there would exist $\eta \in E_i^+(v)$, $i = 1$ or 2, such that $\check{R}_v(\eta, \xi_1^-, \xi_2^+, \xi_1^-) \neq 0$. By Lemma 3, $\chi_i(v) = -\chi_2(v) + 2\chi_1(v) = 0$, a contradiction, since the absolute values of all the positive Lyapunov exponents are bounded below by the constant a of (0). A similar argument applies to each of the remaining terms.

Now, define the vector bundle $\rho: \mathscr{V} = E^+ \oplus E^- \oplus E^+ \oplus E^- \to V$ and consider $\mathscr{N} = \{\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathscr{V}: \check{R}(\xi) = 0\}$. It follows from Lemma 3 that $\check{R}(\xi_2^+, \xi_1^-, \xi_2^+, \xi_1^-) = 0$ for every $v \in \Lambda^-$ so that

$$(E_2^+ \oplus E_1^- \oplus E_2^+ \oplus E_1^-)_v \subset \mathscr{N}_v \quad \text{for each } v \in \Lambda^-.$$

On the other hand, $(*_2)$ can clearly be rephrased as follows (see **[F–K]**, Lemma 3). Given $v \in \Lambda^-$ where $(*_2)$ holds, then for every tangent vector $X \in T_\xi \mathscr{V}$ at $\xi = (\xi_2^+, \xi_1^-, \xi_2^+, \xi_1^-) \in \rho^{-1}(v)$ such that $(D\rho)_\xi X = \xi_2^- \in E_2^-(v) \backslash \{0\}$, we have

$$(D\check{R})_\xi X = \omega_v(\xi) \neq 0.$$

By the Implicit Function Theorem, there exists a neighbourhood $\mathscr{U}_\xi$ of ξ in $\mathscr{V}$ such that the level set $\mathscr{N} \cap \mathscr{U}_\xi$ is a smooth submanifold of codimension 1 embedded in $\mathscr{V}$. Moreover, if $Y \in T_\xi \mathscr{N}$, $(D\check{R})_\xi Y = 0$, so that there does not exist any $Y \in T_\xi \mathscr{N}$ such that $(D\rho)_\xi Y = \xi_2^-(v) \neq 0$. Therefore v is a critical value of the smooth function ρ restricted to $\mathscr{N} \cap \mathscr{U}_\xi$. The union $\bigcup_\xi (\mathscr{N} \cap \mathscr{U}_\xi)$ over all $v \in \Lambda'$ and $\xi \in (E_2^+ \oplus E_1^- \oplus E_2^+ \oplus E_1^-)_v$ such that $(*_2)$ holds is a smooth submanifold of $\mathscr{V}$. The base point projection ρ, restricted to this submanifold, is a smooth map into a manifold of smaller dimension, of which all the points $v \in \Lambda'$ are critical values. It follows from Sard's Theorem that $\lambda(\Lambda') = 0$.

Case III of Table 1 is similar. In fact, this case corresponds to having $\chi_2 < 2\chi_1$, which is shown in **[F–K]** to occur at most in a set of λ-measure zero (if $\omega \neq 0$). □

Let $v \in \mathscr{U} \mapsto L(v) \subset E^\varepsilon(v)$ be a smooth field of k-planes in E^ε defined on an open set $\mathscr{U} \subset V$. Let Ω be the symplectic form on $E = E^+ \oplus E^-$. Define the Ω-complement of L to be the distribution $L^{\perp}$ of linear subspaces of $E^{-\varepsilon}$ of codimension k given by

$$L^{\perp}(v) := \{\eta \in E^{-\varepsilon}(v) : \Omega(\eta, \xi) = 0 \quad \text{for every } \xi \in L(v)\}.$$

Clearly, if L is flow-invariant and smooth on $\mathcal{U}$, then $L^{\perp}$ has the same properties.

Consider now the case when $\dim E^{\varepsilon} = 2$. Let $E^{\varepsilon} = E_1^{\varepsilon} \oplus E_2^{\varepsilon}$, $\varepsilon = +$ or $-$, be the Oseledec decomposition of E^{ε} defined on Λ^{δ} (see Lemma 5). It is an easy consequence of Lemma 3, applied to the nondegenerate form Ω, that

$$(E_1^{\varepsilon})^{\perp} = E_2^{-\varepsilon} \quad (E_2^{\varepsilon})^{\perp} = E_1^{-\varepsilon} \quad \varepsilon = +, -.$$

LEMMA 7. *Assume the same conditions as in Theorem* 1, *and in addition* $\nabla R \neq 0$. *Then, there exist* C^{∞}*-line fields* $v \mapsto L_i^{\varepsilon}(v)$, $v \in V$, *which are flow-invariant and such that for* $v \in \tilde{\Lambda}$ (*see Lemma* 6), $L_i^{\varepsilon}(v) = E_i^{\varepsilon}(v)$.

Proof. To simplify the notation, we assume $\delta = -$. For $v \in V$, consider $L_2^-(v) := \{\eta \in E^-(v) : \omega_v(\eta, \xi_1, \xi_2, \xi_3, \xi_4) = 0$, for every $\xi_1, \xi_3 \in E^+(v)$, $\xi_2, \xi_4 \in E^-(v)\}$. $L_2^-(v)$ is a linear subspace of $E^-(v)$ for each $v \in V$ and can be viewed as the solution set of a system of linear equations on E^- that are smoothly parametrized by $v \in V$. Therefore, there exists a nonempty open set $\mathcal{U} \subset V$ where L_2^- defines a smooth distribution. Since ω and E^{ε} are flow-invariant, $\mathcal{U}$ can be chosen flow-invariant and L_2^- will be flow invariant. In particular $\mathcal{U}$ has full λ-measure. Now, observe that for every $v \in \tilde{\Lambda} \cap \mathcal{U}$, Lemma 6 gives $L_2^-(v) = E_2^-(v)$, so L_2^- is a smooth line field on $\mathcal{U}$ which extends the *fast contracting* direction E_2^-.

Define $L_1^+ := (L_2^-)^{\perp}$. L_1^+ is a flow-invariant, smooth line field on $\mathcal{U}$ and $L_1^+(v) = E_1^+(v)$ for every $v \in \tilde{\Lambda}$.

Now, for $v \in V$ consider

$$Q(v) := \{\eta \in E^+(v) : \omega_v(\xi_1, \eta, \xi_2, \eta, \xi_3) = 0 \quad \text{for every } \xi_1, \xi_2, \xi_3 \in E^-(v)\}$$

$Q(v)$ is the solution set of a system of homogeneous quadratic equations defined on the 2-dimensional space $E^+(v)$. Therefore $Q(v)$ may be the set $\{0\}$, or $E^+(v)$, or one line through the origin or yet a pair of transverse lines intersecting at the origin. Moreover, the system of quadratic equations is smoothly parametrized by $v \in V$ so that there exists a nonempty open set $\mathcal{U}' \subset V$, which can be taken flow-invariant, where Q depends smoothly on v. But for $v \in \tilde{\Lambda} \cap \mathcal{U}'$, we know from Lemma 6 that $Q(v) = E_1^+(v) \cup E_2^+(v)$. Therefore Q defines a pair of transverse smooth line fields on $\mathcal{U}'$ which coincides with $E_1^+(v) \cup E_2^+(v)$ on every $v \in \tilde{\Lambda} \cap \mathcal{U}'$.

Define $\tilde{\mathcal{U}} = \mathcal{U}' \cap \mathcal{U}$. $\tilde{\mathcal{U}}$ is an open set of full λ-measure. On $v \in \tilde{\mathcal{U}} \cap \tilde{\Lambda}$, $L_2^+(v) := (Q(v) \backslash L_1^+(v)) \cup \{0\}$ coincides with $E_2^+(v)$, so that L_2^+ defines a smooth flow invariant line field on $\tilde{\mathcal{U}}$ that extends E_2^+. $L_1^- := (L_2^+)^{\perp}$ is, then, a smooth flow-invariant line field on $\tilde{\mathcal{U}}$ that extends E_1^-.

Next, we show that the line fields L_i^{ε}, $i = 1, 2$, $\varepsilon = +, -$, defined on $\tilde{\mathcal{U}} \subset V$, extend to smooth line fields everywhere on V. For that, it will be enough to find extensions of L_1^{ε} since $L_2^{-\varepsilon} = (L_1^{\varepsilon})^{\perp}$. Note that $L_1^{-\varepsilon}$ is invariant under the holonomy transport along $W^{\varepsilon 0}$ (Lemma 4, § 3). Therefore, Lemma 7 is established after

LEMMA 8. *Assume that the foliations* W^{ε} *and* $W^{\varepsilon 0}$ (*see* § 1) *are* C^{∞} *for* $\varepsilon = +, -$. *Let* $v \in \tilde{\mathcal{U}} \mapsto L(v) \subset E^+(v)$ *be a* C^{∞} *flow-invariant distribution of k-dimensional planes* *defined on an open set $\emptyset \neq \tilde{\mathscr{U}} \subset V$. Assume that L is invariant under the holonomy transport along W^{-0}, in the following sense: If $u, u' \in \tilde{\mathscr{U}}$ are such that $u' \in W^-(u)$, we have $H_{uu'}(u')L(u') = L(u)$ (recall the definitions in § 3). Then, there exists a C^∞ distribution defined everywhere on V and invariant under holonomy transport along W^{-0} that coincides with L on $\tilde{\mathscr{U}}$.*

Proof. Given $v \in V$, define $L(v) := H_{vu}(u)L(u)$, for any $u \in \tilde{\mathscr{U}} \cap W^{-0}(v)$. The first thing to show is that L is well-defined. We note that $\tilde{\mathscr{U}} \cap W^{-0}(v) \neq \emptyset$ since all the leaves of W^{-0} are dense in V (this is true for any Anosov flow possessing a dense orbit. See [A]). Moreover, if u' is another point in $\tilde{\mathscr{U}} \cap W^{-0}(v)$, then $u' \in W^{-0}(u)$, so that

$$H_{vu}(u)L(u) = H_{vu'}(u')H_{u'u}(u)L(u) = H_{vu'}(u')L(u'),$$

and L is therefore independent of the choice of u. L is clearly smooth and invariant along the leaves of W^{-0}. It then suffices to check it is smooth along the leaves of W^+. Let w belong to a sufficiently small neighbourhood $\mathscr{V}$ of v in $W^+(v)$ such that $\mathscr{H}_{vu}^{-1}(w) \in W^+(u) \cap \mathscr{U}$ and $\mathscr{H}_{vu}: \mathscr{V} \to \mathscr{H}_{vu}(\mathscr{V})$ is a diffeomorphism. Then

$$w \in \mathscr{V} \mapsto L(w) = H_{vu}(\mathscr{H}_{vu}^{-1}(w))L(\mathscr{H}_{vu}^{-1}(w))$$

will depend smoothly on w.

End of the proof of Lemma 7. It follows immediately from Lemma 8 that L_1^+ can be extended everywhere on V with the required properties. The same lemma applies to L_1^- by viewing E_1^- as the slow expanding direction for the flow φ_{-t}. Now, take the Ω-complement of L_1^ε to obtain extensions for L_1^ε, $\varepsilon = +, -$. □

We remark that Lemma 7 is sufficient to establish Theorem 2.

It remains to prove that the line fields L_1^ε and L_2^ε, obtained above, are everywhere transverse to each other, and to check the properties claimed for the Lyapunov exponents and the brackets of the line fields.

Proof of Theorem 1. We have shown in Lemma 7 the existence of C^∞ line fields L_i^ε, for $i \in \{1, 2\}$, and $\varepsilon \in \{+, -\}$, such that, for almost every $v \in V$, $L_i^\varepsilon(v) = E_i^\varepsilon(v)$. Here, E_i^ε are the measurable line fields that appear in the Oseledec decomposition of TV,

$$E^\varepsilon = E_1^\varepsilon \oplus E_2^\varepsilon.$$

By Lemma 6, we have

$$\omega(L_{i_1}^\delta, L_{i_2}^{-\delta}, L_{i_3}^\delta, L_{i_4}^{-\delta}, L_{i_5}^\delta) \neq 0 \tag{1}$$

if and only if $(i_1, \ldots, i_5) \in \{(1, 1, 1, 2, 1), (1, 2, 1, 1, 1)\}$. Moreover $\Omega(L_1^\pm, L_2^\mp) \equiv 0$.

Consider the set $\mathscr{A} = \{v \in V: \omega_v \neq 0\}$. $\mathscr{A}$ is a non-empty φ_t-invariant open set, hence it has full λ-measure since the Lebesgue measure is ergodic.

Define the space $\mathscr{M}$ of all φ_t-invariant Borel probability measures on V, equipped with the weak*-topology.

We have: Given any $\mu \in \mathscr{M}$, for μ-almost every $v \in \mathscr{A}$,

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|D\varphi_t|_{L_i^\varepsilon(v)}\| = \varepsilon i \chi_\mu(v), \tag{2}$$

where $\varepsilon \in \{+, -\}$, $i \in \{1, 2\}$, and χ_μ is a φ_t-invariant measurable function on V such that $\chi_\mu \geq a > 0$ (a is the constant that appears in (0)). In fact, this follows from the Oseledec Theorem and Lemma 3, applied to (1).

In particular, we observe that for any point $v \in \mathscr{A}$ that lies on a periodic orbit, (2) is satisfied, since we may consider the invariant probability measure supported on that periodic orbit.

Let us define, for $v \in V$ and $m \in \mathbb{Z}$,

$$F_m(v) := F_m^{\varepsilon,i}(v) := \log \|(D\varphi_m)_v|_{L_i^\varepsilon(v)}\|,$$

where

$$\|(D\varphi_m)_v|_{L_i^\varepsilon(v)}\| = \sup_{\eta \in L_i^\varepsilon(v)-\{0\}} \frac{\|(D\varphi_m)_v\eta\|}{\|\eta\|} = \frac{\|(D\varphi_m)_v\xi\|}{\|\xi\|} \quad \text{for any } \xi \in L_i^\varepsilon(v)-\{0\}$$

(recall L_i^ε is one-dimensional).

It immediately follows that $\{F_m : m \in \mathbb{Z}\}$ is an *additive cocycle*, that is,

$$\begin{aligned} F_{m+n}(v) &= \log \frac{\|(D\varphi_{m+n})_v\xi\|}{\|\xi\|} = \log \frac{\|(D\varphi_m)_{\varphi_n(v)}(D\varphi_n)_v\xi\|}{\|(D\varphi_n)_v\xi\|} + \log \frac{\|(D\varphi_n)_v\xi\|}{\|\xi\|} \\ &= F_m(\varphi_n(v)) + F_n(v). \end{aligned}$$

Therefore, due to Birkhoff's Ergodic Theorem, if μ is an ergodic measure in $\mathscr{M}$, for μ-almost every $v \in V$, the limit of

$$\frac{F_m(v)}{m} = \frac{1}{m} \sum_{i=1}^{m-1} F_1(\varphi^{m-1}(v)) \quad \text{for } m \to \infty,$$

exists and equals $\int_V F_1 \, d\mu$.

In particular, if $v \in \mathscr{A}$ is any point lying on a periodic orbit of φ_t and $\mu \in \mathscr{M}$ is the measure supported on that orbit, then there exists $\chi(v) \geq a > 0$ so that

$$\int_V F_1^{\varepsilon i} \, d\mu = \varepsilon i \chi(v) \quad \varepsilon \in \{+, -\},\ i \in \{1, 2\}. \tag{3}$$

We observe that the positive functions $\mu \mapsto \mathscr{F}^{\varepsilon,i}(\mu) = \int_V F_1^{\varepsilon,i} \, d\mu$, defined on $\mathscr{M}$ are continuous with respect to the weak*-topology.

Any ergodic measure $\mu \in \mathscr{M}$ can be obtained as the limit of a sequence of measures $\mu_n \in \mathscr{M}$ supported on periodic orbits that are contained in any given invariant dense open set. This is a corollary of the *Specification Theorem* for Anosov flows [**B**], and can be shown as follows. Let $\mu \in \mathscr{M}$ be ergodic. Then, it follows from Birkhoff's Ergodic Theorem that for μ-almost every $v \in V$, and for every continuous function f on V,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\varphi_t(v)) \, dt = \int_V f \, d\mu.$$

Choose such a point v and pick any $v_0 \in \mathscr{A}$. Now, we can apply the Specification Theorem to produce a closed orbit that passes near v_0 (so that it is contained in $\mathscr{A}$) and follows the orbit segment of v of length T for most of the time, within a small distance ε. With an appropriate choice of ε and T, it is easy to verify that the invariant probability measure supported on that closed orbit will approximate μ arbitrarily well in the weak*-topology.

Therefore, due to Lemma 3, given any ergodic measure $\mu \in \mathcal{M}$ and for μ-almost every $v \in V$, there exists $\chi_\mu \geq a > 0$ so that (3) also holds for μ. In fact, by approximating μ by a sequence of measures $\mu_n \in \mathcal{M}$ supported on periodic orbits in $\mathcal{A}$, we obtain

$$\frac{\mathcal{F}^{\varepsilon,2}(\mu)}{\mathcal{F}^{\varepsilon,1}(\mu)} = \lim_{n\to\infty} \frac{\mathcal{F}^{\varepsilon,2}(\mu_n)}{\mathcal{F}^{\varepsilon,1}(\mu_n)} = 2. \tag{4}$$

Let us prove now that $L_1^\varepsilon \neq L_2^\varepsilon$ everywhere. The sets $\mathcal{O}^\varepsilon = \{v \in V\colon L_1^\varepsilon(v) = L_2^\varepsilon(v)\}$, for $\varepsilon = +$ and $-$, are compact and flow-invariant. If at least one of them is not empty, it carries a flow-invariant ergodic measure μ for which $\mathcal{F}^{\varepsilon,2}(\mu)/\mathcal{F}^{\varepsilon,1}(\mu) = 1$, thus contradicting (4). Therefore, $\mathcal{O}^+ = \mathcal{O}^- = \emptyset$.

It remains to show that $L_1^\varepsilon \subset [L_2^\varepsilon, L_1^{-\varepsilon}]$ on a nontrivial invariant open set, for some ε. Consider smooth vector fields $\xi_i^\pm$, for $i \in \{1, 2\}$, such that $\xi_i^\pm(v) \in L_i^\pm(v)$ for all v. Since $L_i^\pm = E_i^\pm$ almost everywhere, and due to Lemma 3, we have, for any sign ε and any vector field $\eta \in E^\varepsilon$,

$$\check{R}(\xi_1^\varepsilon, \xi_1^{-\varepsilon}, \xi_2^\varepsilon, \xi_1^{-\varepsilon}) \equiv 0 \quad \text{and} \quad \check{R}(\eta, \xi_1^{-\varepsilon}, \xi_2^\varepsilon, \xi_1^{-\varepsilon}) \equiv 0.$$

Therefore

$$\begin{aligned}
\omega(\xi_1^{-\varepsilon}, \xi_1^\varepsilon, \xi_1^{-\varepsilon}, \xi_2^\varepsilon, \xi_1^{-\varepsilon}) &= \xi_1^{-\varepsilon}\check{R}(\xi_1^\varepsilon, \xi_1^{-\varepsilon}, \xi_2^\varepsilon, \xi_1^{-\varepsilon}) - \check{R}(\nabla_{\xi_1^{-\varepsilon}}\xi_1^\varepsilon, \xi_1^{-\varepsilon}, \xi_2^\varepsilon, \xi_1^{-\varepsilon}) \\
&\quad - \check{R}(\xi_1^\varepsilon, \nabla_{\xi_1^{-\varepsilon}}\xi_1^{-\varepsilon}, \xi_2^\varepsilon, \xi_1^{-\varepsilon}) - \check{R}(\xi_1^\varepsilon, \xi_1^{-\varepsilon}, \nabla_{\xi_1^{-\varepsilon}}\xi_2^\varepsilon, \xi_1^{-\varepsilon}) \\
&\quad - \check{R}(\xi_1^\varepsilon, \xi_1^{-\varepsilon}, \xi_2^\varepsilon, \nabla_{\xi_1^{-\varepsilon}}\xi_1^{-\varepsilon}) \\
&= -2\cdot\check{R}(\xi_1^\varepsilon, \xi_1^{-\varepsilon}, \xi_2^\varepsilon, \nabla_{\xi_1^{-\varepsilon}}\xi_1^{-\varepsilon}) - \check{R}(\xi_1^\varepsilon, \xi_1^{-\varepsilon}, \nabla_{\xi_1^{-\varepsilon}}\xi_2^\varepsilon, \xi_1^{-\varepsilon}),
\end{aligned}$$

where for the last step we also used the symmetries of $\check{R}$ given in Lemma 2. Therefore, if we choose $\varepsilon = \delta$ (see Lemma 6), we have

$$\omega(L_1^{-\delta}, L_1^\delta, L_1^{-\delta}, L_2^\delta, L_1^{-\delta}) \neq 0$$

on some invariant nontrivial open set, so that $L_2^{-\delta} \subset \nabla_{L_1^{-\delta}} L_1^{-\delta}$ or $L_1^\delta \subset \nabla_{L_1^{-\delta}} L_2^\delta$ on that open set. But

$$\Omega(\nabla_{\xi_1^{-\delta}}\xi_2^\delta, \xi_1^{-\delta}) = \xi_1^{-\delta}\Omega(\xi_2^\delta, \xi_1^{-\delta}) - \Omega(\xi_2^\delta, \nabla_{\xi_1^{-\delta}}\xi_1^{-\delta}) = -\Omega(\xi_2^\delta, \nabla_{\xi_1^{-\delta}}\xi_1^{-\delta}),$$

so that both inclusions should occur, since $\Omega(L_i^\pm, L_j^\mp) \neq 0$ if and only if $i = j$. Therefore, since $\nabla E^\pm \subset E^\pm$,

$$L_1^\delta \subset \nabla_{L_1^{-\delta}} L_2^\delta \subset \nabla_{L_1^{-\delta}} L_2^\delta - \nabla_{L_2^\delta} L_1^{-\delta} \subset [L_2^\delta, L_1^{-\delta}]. \qquad \square$$

Proof of Theorem 2. In [**K**], M. Kanai considered the following setting. Denote by $P = \tilde{V}/\{\phi_t\}$ the space of orbits of the lift of the geodesic flow to the universal cover $\tilde{V}$ of V, where V is now the unit tangent bundle of a negatively curved manifold M, as in Theorem 2. On P, he introduced an affine connection, let us call it ∇', which can be thought of as the restriction of ∇ to the bundle E (see [**F–K**] for details). He proved that, if ∇' is a locally symmetric affine connection, i.e. if $\nabla' R' \equiv 0$, where R' is the curvature tensor of ∇', then the conclusion of Theorem 2 holds. Therefore, it will suffice to show that the tensor field ω vanishes in this case.

Suppose $\omega \neq 0$. According to Theorem 1, there exists a smooth line field $L = L_1^+ \subset E^+$ defined everywhere on V.

Let S be a closed surface diffeomorphic to the 2-sphere, embedded in the universal cover $\tilde{M}$ of M. Denote by $\nu(x)$ the inward, say, unit normal vector to S at $x \in S$ and let $\rho : \tilde{V} \to \tilde{M}$ denote the base point projection. The differential of ρ at $v \in \tilde{V}$ defines an isomorphism between $E^+(v)$ and the orthogonal complement of v, $v^\perp \subset T_{\rho(v)}\tilde{M}$. Therefore, for each $x \in S$, $(D\rho)_{\nu(x)}: E^+(\nu(x)) \to T_x S$ is a linear isomorphism and

$$x \in S \mapsto (D\rho)_{\nu(x)} L(\nu(x)) \subset T_x S$$

defines a continuous line field tangent to S, a topological impossibility. Therefore, we must have $\omega \equiv 0$. □

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